

Understanding and modelling nonlinear mechanisms in flow separation

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Fluid flows, governed by the Navier-Stokes equations, are spatio-temporal systems that often exhibit complex behaviors whose modelling and physical understanding are still the object of ongoing research. The challenges that arise when studying such systems are typically related to their high dimensionality and their nonlinear nature. The latter, in the form of an energy-conserving, quadratic nonlinearity ($\mathbf{u} \cdot \nabla \mathbf{u}$), is responsible for the interaction and energy exchange between different time- and length-scales. Said nonlinear interaction is especially relevant in flow separation, which is a common phenomenon that is typically induced by adverse pressure gradients and geometric discontinuities and leads to energy loss and increased drag. Vortical structures of different length- and time-scales usually form at and downstream of the separation point, and their nonlinear interaction may give rise to complex phenomena such as "lock-on". In the interest of controlling this type of flow, it is therefore necessary to understand the physics behind such nonlinear mechanism.

Modelling the linearities: Resolvent Analysis

Resolvent analysis is a tool that finds its roots in linear, modern control theory and leads to an input-output representation of a given system. It is common, in incompressible fluid mechanics, to consider the fluctuations of the velocity and pressure fields about a nominal base flow (often times taken to be the temporal mean flow) and cast the discretized system of equations as follows:

$$\mathbf{M} \frac{d}{dt} \mathbf{q} = \mathbf{A} \mathbf{q} + \mathbf{f}(\mathbf{q}) \quad (1)$$

where $\mathbf{q} = [u, v, w, P]^T \in \mathbb{R}^N$ is the state vector of fluctuating quantities, $\mathbf{A} \in \mathbb{R}^{N \times N}$ is the Navier-Stokes operator linearized about the temporal mean flow, \mathbf{M} is a matrix of weights and $\mathbf{f}(\mathbf{q}) \in \mathbb{R}^N$ is a nonlinear function of \mathbf{q} . In the framework developed by Jovanovic and Bamieh ([1]) $\mathbf{f}(\mathbf{q})$ is treated as an arbitrary input, $\mathbf{u} = \hat{\mathbf{u}} e^{i\omega t}$. Moreover, assuming global forcing, the system can be cast in the following input-output form:

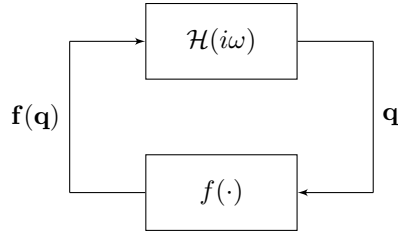
$$\hat{\mathbf{q}} = (i\omega \mathbf{M} - \mathbf{A})^{-1} \hat{\mathbf{u}} \quad (2)$$

where $\mathcal{H}(i\omega) = (i\omega \mathbf{M} - \mathbf{A})^{-1}$ is the input-output (resolvent) operator. The resolvent can then be SVD-decomposed to obtain the output modes and the forcing modes at a fixed frequency, ω . Although the output modes can also be computed from data, using Spectral Proper Orthogonal Decomposition (SPOD) ([4]), resolvent analysis allows to compute both the output and the forcing modes at once from the mere knowledge of the time-mean flow, without having to resort to expensive forward and adjoint CFD simulations. Furthermore, it is often the case that the resolvent operator is low rank ([2], [3]), meaning that its action can be accurately captured by the first few output and forcing modes along with the corresponding singular values, and this is of course advantageous inasmuch as it allows for a reduced-order representation of the dynamics of the system at the frequency under consideration. However, while the resolvent is indicative of the linear dynamics of the fluctuation field about the time-mean flow, the difficulties in the context of a frequency-domain analysis are tied to the nonlinear forcing term, $\mathbf{f}(\mathbf{q})$.

Modelling the nonlinearities: Describing Function Analysis

First and foremost, the nonlinear operator in the Navier-Stokes equation is static and energy-conserving. Stationarity is indicative of the fact that the operator is frequency-independent, while energy-conservation, a special case of passivity ([5]), refers to the fact that it does not produce nor

dissipate energy in the system. The nonlinearity, in fact, is only responsible for the redistribution of energy across all frequencies, and its contribution to the dynamics of the system is shown in the block-diagram representation below:



Nonlinear systems in feedback form can be studied using describing function analysis, which is a tool rooted in nonlinear feedback systems and originally used to model the effect of static nonlinearities such as saturations and dead-zones ([5]). An extended version of describing function analysis was successfully tested on a 3-dimensional model problem whose nonlinearity is similar to the nonlinearity in the Navier-Stokes equation (i.e. static, quadratic and energy-conserving). A future direction will be to extend this approach to high-dimensional systems in order to (potentially) incorporate it in existing frequency-domain analyses of fluid flows.

References

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