Definition of Stellarator Equilibrium with Minimal Unknowns and its use for Numerical Applications

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Controlled fusion power has long been sought after as the panacea for our global energy and climate crises. Substantial progress has been made towards achieving this dream, but several formidable challenges remain. The ability to efficiently calculate equilibria in fusion devices is critical to deepen our physical understanding of plasma science, improve diagnostic capabilities, avoid disruptions that terminate plasma confinement, and enable real-time control systems. One of the leading types of fusion devices is the stellarator, which exploits three-dimensional magnet geometries to provide advantages over axisymmetric, two-dimensional designs. While three-dimensional equilibria construction techniques already exist, the modern methods are neither accurate nor quick enough for commercial fusion power.

Equilibrium refers to the full state of the plasma for a given configuration: the shapes of flux surfaces, the magnetic field vectors, and profiles of all other quantities of interest such as the plasma pressure p and rotational transform ι. The equilibrium that a steady-state fusion reactor operates at will determine its confinement performance and stability properties. Equilibria construction is essential both when analyzing the designs of new devices and when optimizing the performance of existing experiments. For large time and space scales, the plasma can be modeled to first order as a single, perfectly conducting fluid. This ideal magnetohydrodynamic (MHD) approximation simplifies the equilibrium condition to three equations [1]:

$$\mathbf{J} \times \mathbf{B} = \nabla p \tag{1}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{2}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{3}$$

where $\mathbf{J}$ is the plasma current, $\mathbf{B}$ is the magnetic field, and μ_0 is the magnetic constant.

The MHD equilibria equations look deceptively simple but can be remarkably difficult to solve. Analytic equations are rarely applicable, and generally equilibria must be calculated numerically. The most widely used MHD equilibria code, VMEC, is based on the energy principle, since the equilibrium corresponds to a minimum energy state. A system of nonlinear, second-order, hyperbolic differential equations are solved using an explicit scheme until the kinetic energy is eliminated, the total energy is minimized, and an equilibrium is obtained [2]. Nested flux surfaces are assumed to ensure the magnetic field is divergence-free, but the minimum energy solution is not guaranteed to satisfy the equilibrium force balance. This code becomes computationally expensive for three-dimensional geometries, and convergence is slow for the desired levels of accuracy. As modern stellarator experiments continue to show promising results, it is becoming increasingly important to advance our three-dimensional equilibria construction methods.

A novel technique for solving ideal MHD equilibria is presented in this address. This method uses a Poincare section approach instead of the energy principle to determine the validity of an equilibrium solution. The plasma is modeled as a system of nonlinear, second-order, ordinary differential equations (ODEs), where the state variables represent the locations of the flux surfaces on a Poincare section, the time variable represents the toroidal angle coordinate, and the ODEs were derived from the MHD equilibrium equations. For a stellarator, the equilibrium Poincare section will show the same flux surfaces after following the magnetic field lines for a full transit in the toroidal direction. Provided an initial guess for the equilibrium, a solution can be found by iteratively perturbing the system variables proportional to their integration errors until the periodic boundary condition is reached. Mathematically, this approach is analogous to casting the equilibrium equations into a boundary value problem (BVP). From a physics perspective, this technique follows magnetic field lines to compare Poincare sections, and is not directly concerned with minimizing the total energy of the system.

This proposed method could provide several advantages over the conventional approach. First and foremost, the equilibrium force balance is added as a constraint into the ODE system. Unlike the energy principle technique which is subject to getting caught in local minima that do not correspond to the desired equilibrium, any solution found by this Poincare section approach would be guaranteed to satisfy all the MHD equilibrium conditions. Furthermore, adding all the equilibrium equations as constraints reduces the system to the minimum number of dimensions. The full set of variables that represent an equilibrium are condensed onto a single Poincare section, and there are the same number of equations as unknowns. This reduced dimensionality greatly improves the computational efficiency of calculations, since the complexity of an optimization problem scales with the number of variables. Consequently, this method could result in a code that is significantly faster than the current standard, perhaps even being able to run in real-time.

Although fusion devices are described well by the toroidal coordinate system (R, Z, ϕ), it is often convenient to work with a curvilinear coordinate system (ρ, θ, ζ) that makes the magnetic field lines appear straight. The magnetic field can be written in contravariant form $\mathbf{B} = B^\theta \mathbf{e}_\theta + B^\zeta \mathbf{e}_\zeta$ to assume nested flux surfaces and satisfy the absence of magnetic monopoles (3). Substituting this form of the magnetic field into the equilibrium force balance equation (1) using Ampere's Law (2) yields a vectoral expression for the force balance error: $\mathbf{F} = F_\rho \nabla\rho + F_\beta \beta = 0$. F_ρ and F_β are two independent components and must both vanish in equilibrium [2]. With the general Clebsch form of the magnetic field $\mathbf{B} = \frac{\partial_\rho \Psi}{2\pi\sqrt{g}}(\iota \mathbf{e}_\theta + \mathbf{e}_\zeta)$ and definition of the covariant basis vectors $\mathbf{e}_i = [\partial_i R, \partial_i Z, \partial_i \phi]^T$, the magnetic field components can be written in terms of the coordinates R and Z and their spatial derivatives [3]. The two equations $F_\rho = F_\beta = 0$ can then be solved for $\partial_{\zeta\zeta} R$ and $\partial_{\zeta\zeta} Z$, which gives the system of ODEs for the state variables $\mathbf{x}(\zeta) = [R, Z, \partial_\zeta R, \partial_\zeta Z]^T$. Combining these ODEs with the toroidally periodic boundary condition $\mathbf{x}(0) = \mathbf{x}(2\pi)$ provides a well-posed BVP that can be solved for an equilibrium Poincare section.

This presentation covers the detailed derivation of the BVP formulation from the ideal MHD equilibrium equations. The equations are validated by comparison to results from VMEC for stellarator equilibria with fixed boundaries. Some details of the computational methods used in the new code are discussed. The continued development of this Poincare section approach will require the design of an optimization technique to converge an initial guess towards the equilibrium solution. Other major areas of future work include reformulating the problem for free-boundary instead of fixed-boundary solutions, and discovering a way to allow for magnetic islands when it is not reasonable to assume that nested flux surfaces exist.

References

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