Cascade Centrality in Two-layer Multiplex Networks

Yaofeng (Desmond) Zhong

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1 Background

1.1 A linear threshold model of cascades in networks

We investigate cascades in networks by a linear threshold model. We encode the network topology in a graph $G = (V, E)$. Each agent corresponds to a node in the set $V = \{1, 2, 3, ..., n\}$. $E \subseteq V \times V$ is the set of edges. We consider a undirected unweighted graph. Each agent in G has a state which takes one of the two values - on (active) and off (inactive). Initially, all agents are switched off except an initial active set S_0. All agents choose individual thresholds randomly and independently drawn from a uniform distribution on the interval $[0, 1]$. In the following process, once the proportion of neighbours that switch exceeds one's threshold, it switches on. Once an agent switches, it remains switched forever. This process propagates through the network. Other dynamical models on networks regarding cascades are shown in [1] and [2].

1.2 Cascade centrality

The cascade centrality[4] of node v is a property of v which reflects its ability of spreading information through the network. Cascade centrality is defined as the expected number of active nodes at the end of the linear threshold process given v is the only initial active node.

1.3 Multiplex networks

A multiplex network $\mathcal{G}$ is a family of $m \in \mathbb{N}$ undirected unweighted graphs $G_1, ..., G_m$. Each graph $G_k = (V, E^k), 1 \leq k \leq m$ is referred to as a layer of the multiplex network. The node set $V = 1, 2, 3, ..., n$ is the same in all layers. The edge set of layer k is $E^k \subseteq V \times V$ can be different in different layers. In my research, I start with two-layer multiplex networks. Here I assume both G_1 and G_2 are connected graphs. The multiplex network is a special type of the multilayer network.[3]

2 Goal

Lim et al.[4] show a simple and analytically tractable way of calculating cascade centrality. My goal is to generalize cascade centrality to multiplex networks, find a systematic and analytically tractable way to calculate it and derive analytical results which can give us insights into cascades in multiplex networks. In multiplex networks, cascade centrality of v is also defined as the expected number of active nodes at the end of the linear threshold process given v is the only initial active node.

3 Method

The key point in the calculation of cascade centrality of nodes in single layer networks is to determine the probability of switching of each node in the network. Lim et al.[4] leverage results of Kempe et al.[5]. Kempe et al.[5] proves that the linear threshold process produces the same probability distribution in the network as the live-edge model.

3.1 Live-edge Model

For a certain node v, denote the set of its neighbours as N_v and the number of neighbours, i.e. its degree, as d_v. An edge between $u \in V$ and $v \in V$ is denoted as E_{uv}.

In the live-edge model, we transform G into an directed graph by treating each edge E_{uv} in G as two directed edges, one coming from u to v and the other from v to u. We let node v randomly select one edge out of all incoming edges. All edges coming into the node have equal chances to be selected, which is $1/d_v$. The selected edge will be labeled to be "live", while the rest of them will be labeled to be "blocked". Every node in the network follows the same procedure so that in the end, every directed edge will be either live or blocked. If there is a directed path from some node in S_0 to v consisting entirely of live edges (it is defined as a live-edge path), then we say v is reachable from S_0 via live-edge paths.

3.2 Equivalence of the linear threshold model and the live edge model

The two models are both probabilistic models. In the linear threshold model, random variables are the thresholds of agents. At the end of the process each agent will have a probability of being active given S_0. In the live-edge model, random variables are the live edges that nodes select. At the end of the process, each agent will have a probability of being reachable from S_0 via live-edge paths. Kempe et al.[5] prove that the two probability distributions are the same. Directly computing the probability distribution of linear threshold process is not easy since it can only be run iteratively. Thus we turn to calculate the probability distribution of the live-edge model. I seek a similar equivalence in the multiplex context.

3.3 Linear threshold model in two-layer multiplex network

Each agent v chooses a threshold μ_{v1} in layer 1 and a threshold μ_{v2} in layer 2. All thresholds are randomly and independently drawn from a uniform distribution on the interval $[0, 1]$. Since a node need to combine two inputs to make a decision, we propose two protocols:

- **Protocol OR**: a node v will switch if either portion of active neighbours in layer 1 exceeds μ_{v1} or portion of active neighbours in layer 2 exceeds μ_{v2};
- **Protocol AND**: a node v will switch if both portion of active neighbours in layer 1 exceeds μ_{v1} and portion of active neighbours in layer 2 exceeds μ_{v2}.

3.4 Live-edge path model in two-layer multiplex networks

We let node v randomly select one edge in each layer out of all incoming edges in that layer. All edges coming into the node in the same layer have equal chances to be selected. The selected edge will be labeled to be "live", while the rest of them will be labeled to be "blocked". Every node in the network follows the same procedure except nodes in the initial active set S_0. All of the incoming edges of nodes in S_0 will be labeled as "blocked". In the end, every directed edge will be either live or blocked. We then define a live-edge tree.

Definition 3.1. Given a selection of live edges and the initial active set, a *live-edge tree* associated with agent v is a full binary tree that satisfies

1. Every node in the tree corresponds to an agent in the multiplex network $\mathcal{G}$. The root node corresponds to agent v;
2. For each parent p in the tree, p's left child is the agent that p's live edge in layer 1 connects to. p's right child is the agent that p's live edge in layer 2 connects to.

Now we propose two reachability definitions:

- **Reachability OR**: a node v is reachable via a selection of live edges if the live-edge tree associated with v has at least one finite branch. In other words, at least one branch ends with a node in S_0;
- **Reachability AND**: a node v is reachable via a selection of live edges if all branches of the live-edge tree associated with v are finite. In other words, all branches end with nodes in S_0.

4 Results

I prove that the probability distribution of switching by running linear threshold model under Protocol OR/AND given S_0 is the same as the probability distribution of reachability from S_0 defined by Reachability OR/AND. Then I provide a systematic way of calculating this probability distribution. I also provide an example of Protocol OR as well as an example of Protocol AND, and show a way of calculating the probability manually.

5 Future Work

Since the general way of computing the probability distribution is not time efficient. We aim to show some bound or order of the probabilities distribution of different networks. We can also generalize our method in multiplex networks with more than two layers.

References

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